

# Endoscopic lifts to the Siegel modular threefold related to Klein's cubic threefold

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*Dedicated to Professor Takayuki Oda on his 60-th birthday.*

**Abstract.**— Let  $\mathcal{A}_{11}^{\text{lev}}$  be the moduli space of  $(1, 11)$ -polarized abelian surfaces with a canonical level structure. Let  $\chi$  be a primitive character of order 5 with conductor 11. In this paper we construct five endoscopic lifts  $\Pi_i, 0 \leq i \leq 4$  from two elliptic modular forms  $f \otimes \chi^i$  of weight 2 and  $g \otimes \chi^i$  of weight 4 with complex multiplication by  $\mathbb{Q}(\sqrt{-11})$  such that  $\Pi_{i\infty}$  gives a non-holomorphic differential form on  $\mathcal{A}_{11}^{\text{lev}}$  for each  $i, 0 \leq i \leq 4$ . Then their spinor  $L$ -functions are of form  $L(s-1, f \otimes \chi^i)L(s, g \otimes \chi^i)$  such that  $L(s, g \otimes \chi^i)$  does not appear in the  $L$ -function of  $\mathcal{A}_{11}^{\text{lev}}$  for any  $i, 0 \leq i \leq 4$ . The existence of such lifts is motivated by the computation of the  $L$ -function of Klein's cubic threefold which is a birational smooth model of  $\mathcal{A}_{11}^{\text{lev}}$ .

## 1. Introduction

Let  $\mathcal{A}_{11}^{\text{lev}}$  be the moduli space of  $(1, 11)$ -polarized abelian surfaces with a canonical level structure (see [13] for details). It is obtained as a quotient of the Siegel upper half plane  $\mathbb{H}_2$  by the arithmetic subgroup  $K(11)^{\text{lev}}$  in the symplectic group  $\text{Sp}_2(\mathbb{Z}) \subset \text{GL}_4(\mathbb{Z})$  (see Section 3 for the definition of  $K(11)^{\text{lev}}$ ). The congruence subgroup

$$\Gamma(11) := \{\gamma \in \text{Sp}_2(\mathbb{Z}) \mid \gamma \equiv 1_4 \pmod{11}\} \subset \text{Sp}_2(\mathbb{Z})$$

is a normal subgroup of  $gK(11)^{\text{lev}}g^{-1}$  where  $g = \text{diag}(1, 1, 11, 11)$ . Then we have the Galois covering  $\pi : S_{\Gamma(11)} := \Gamma(11) \backslash \mathbb{H} \rightarrow \mathcal{A}_{11}^{\text{lev}}$  of moduli spaces with the Galois group  $G := gK(11)^{\text{lev}}g^{-1}/\Gamma(11)$ . Since  $\Gamma(11)$  is torsion free,  $S_{\Gamma(11)}$  is a quasi-projective smooth variety. On the other hand,  $K(11)^{\text{lev}}$  has many torsion points, so  $\mathcal{A}_{11}^{\text{lev}}$  is a quasi-projective variety but it has many quotient singularities. We can view these moduli spaces as varieties defined over  $\mathbb{Q}$  if we consider the level structure étale locally (cf. [22]). Note that it is easy to see that canonical models of these varieties are both defined over  $\mathbb{Q}(\zeta_{11^2})$ . This fact follows from the moduli interpretation and a basic knowledge of Weil pairing. In [9], Mark Gross and Popescu determined a birational smooth model  $X$  of  $\mathcal{A}_{11}^{\text{lev}}$  by a significant method. More precisely they constructed a birational map  $\mathcal{A}_{11}^{\text{lev}} \rightarrow X$  over  $\mathbb{Q}$  where  $X$  is defined by a simple equation in projective space  $\mathbb{P}_{\mathbb{Q}}^4$ :

$$X : x_0^2x_1 + x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_0 = 0.$$

The variety  $X$  is called Klein's cubic threefold. Since  $X$  is a smooth cubic threefold (hence a Fano threefold), it has Hodge numbers

$$h^{0,0} = 1, h^{1,0} = h^{0,1} = h^{2,0} = h^{0,2} = h^{3,0} = h^{0,3} = 0, h^{1,1} = 1, h^{2,1} = h^{1,2} = 5. \quad (1.1)$$

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In particular, we see that the local  $L$ -factor of  $X$  is of degree 10.

In this paper we first compute the  $L$ -function of  $X$ . As a result we have:

**THEOREM 1.1.** *Let  $\ell$  be a prime number. Let  $f$  be the elliptic modular form of weight 2 for  $\Gamma_0(11^2)$  with complex multiplication by the ring of integers of  $\mathbb{Q}(\sqrt{-11})$ . Let  $\chi$  be a primitive character of order 5 with conductor 11. Then we have  $L(s, H_{\text{et}}^3(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_{\ell})) = \prod_{i=0}^4 L(s-1, f \otimes \chi^i)$ . In particular, the left hand side is independent to any choice of  $\ell$ .*

By Theorem 1.1 and the Hodge type of  $X$ , it is quite natural to predict the existence of non-holomorphic differential forms on  $\mathcal{A}_{11}^{\text{lev}}$  of Hodge type  $(2, 1)$  of which corresponding automorphic forms on  $\text{GSp}_2(\mathbb{A})$  are liftings related to the elliptic modular form  $f$  (we will discuss on the  $(1, 1)$ -part in another paper). In a similar situation, in [28], authors treated a unique holomorphic differential 3-form on some Siegel threefold. Since the space of holomorphic 3-forms is a birational invariant, we can study this form by using an explicit birational model of the Siegel threefold in [28]. In this paper we treat non-holomorphic differential forms. However their space is not a birational invariant. Therefore we cannot directly construct non-holomorphic differential forms on  $\mathcal{A}_{11}^{\text{lev}}$  from them on  $X$ . Note that since  $H^{3,0}(X) = 0$ , there does not exist any holomorphic differential 3-form on  $\mathcal{A}_{11}^{\text{lev}}$  which extends to one on any birational smooth model of  $\mathcal{A}_{11}^{\text{lev}}$ . So this case will give a first and fascinating example to spur authors on an explicit construction of non-holomorphic differential forms of which corresponding automorphic forms on  $\text{GSp}_2(\mathbb{A})$  are liftings related to the elliptic modular form  $f$ .

We now explain the second main result. Let  $\mu$  be the größencharacter of  $K = \mathbb{Q}(\sqrt{-11})$  associated to  $f$  and  $g$  be the elliptic modular form associated to  $\mu^3$ . Note that  $g$  is of weight 4 and of level  $11^2$ . Then we have:

**THEOREM 1.2.** (Theorem 3.2) *There are irreducible, cuspidal, automorphic, globally generic representations  $\Pi_i, 0 \leq i \leq 4$  of  $\text{GSp}_2(\mathbb{A})$  with  $\Pi_{i,\infty}$  being a discrete series whose Blattner parameter is  $(3, -1)$ . Each  $\Pi_i$  satisfies the following properties:*

- (i) *There is a non-zero right  $K(11)_{\mathbb{A}}^{\text{lev}}$ -invariant automorphic form  $F_i \in \Pi_i$ .*
- (ii) *The spinor  $L$ -function of  $\Pi_i$  is  $L(s, \pi_f \otimes \chi^i) L(s, \pi_g \otimes \chi^i)$ , where  $\pi_f$  (resp.  $\pi_g$ ) is the unitary irreducible automorphic representation attached to  $f$  (resp.  $g$ ). (see Novodvorsky [24] for the definition of the spinor  $L$ -function of a generic representation).*

The strategy of the construction of  $\Pi_i$  is as follows. In our case, by combining several facts, we guess that  $\Pi_i$  is a weak endoscopic lift in the sense of [44]. By results of Kudla, Rallis, and Soudry [17], and Roberts [32], we know that such  $\Pi_i$  is given by a  $\theta$ -lift  $\Theta(\pi_1 \boxtimes \pi_2)$  of a pair  $(\pi_1, \pi_2)$  of two irreducible cuspidal automorphic representations of  $\text{GL}_2(\mathbb{A})$  (Here we identify  $(\pi_1, \pi_2)$  with an automorphic representation of  $\text{GSO}_{2,2}(\mathbb{A})$ ). It is natural to guess that  $\pi_1$  should be  $\pi_f \otimes \chi^i$ . We also have to find a candidate of  $\pi_2$ . After trial and error (but there is no precise evidence) we decide  $\pi_2 = \pi_g \otimes \chi^i$  and by choosing a suitable Schwartz-Bruhat function for the  $\theta$ -lift, realize a non-zero right  $K(11)_{\mathbb{A}}^{\text{lev}}$ -invariant vector  $F_i \in \Theta((\pi_f \otimes \chi^i) \boxtimes (\pi_g \otimes \chi^i))$  in Theorem 1.2.

We should discuss a comparison of differential forms and  $L$ -functions between  $X$  and  $\mathcal{A}_{11}^{\text{lev}}$ . Let  $H_{\text{cusp}}^3(\mathcal{A}_{11}^{\text{lev}}, \mathbb{C})$  be the cuspidal part of the Betti cohomology  $H^3(\mathcal{A}_{11}^{\text{lev}}, \mathbb{C})$  (see Section 4). Then by combining above two theorems, we have:

**THEOREM 1.3.** *The notations being as above. Let  $E_i$  be the elliptic curve attached to  $f \otimes \chi^i$  for each  $i \in \{0, 1, 2, 3, 4\}$ . We fix a non-zero holomorphic 1-form  $\omega_i$  on  $E_i$ .*

(i) There exists a linear map  $H_{\text{dR}}^3(X_{\mathbb{C}}) \simeq \bigoplus_{0 \leq i \leq 4} H_{\text{dR}}^3(E_{i\mathbb{C}} \times \mathbb{P}_{\mathbb{C}}^2) \longrightarrow H_{\text{cusp}}^3(\mathcal{A}_{11}^{\text{lev}}, \mathbb{C})$  which is injective

and preserves the Hodge structure. Here the second map is given by  $\omega_i \otimes \Lambda \mapsto F_{i\infty}$ ,  $\bar{\omega}_i \otimes \Lambda \mapsto \overline{F_{i\infty}}$  where  $\Lambda$  is a generator of  $H^{1,1}(\mathbb{P}_{\mathbb{C}}^2)$ .

(ii) for any  $i$ , the  $L$ -function of  $\Pi_i$ , more precisely  $\pi_g \otimes \chi^i$  does not contribute to the  $L$ -function of the parabolic cohomology  $H_{\text{et},!}^3(\mathcal{A}_{11}^{\text{lev}}, \mathbb{Q}_{\ell})$  defined by the image of the natural map from the  $\ell$ -adic étale cohomology with compact support  $H_{\text{et},c}^3(\mathcal{A}_{11}^{\text{lev}}, \mathbb{Q}_{\ell})$  to  $H_{\text{et}}^3(\mathcal{A}_{11}^{\text{lev}}, \mathbb{Q}_{\ell})$ .

Recall that  $\mathcal{A}_{11}^{\text{lev}}$  is a singular variety. So we do not know a priori whether  $H_{\text{et},!}^3(\mathcal{A}_{11}^{\text{lev}}, \mathbb{Q}_{\ell})$  is pure of weight 3. We check this as follows. With the notations of the beginning of the introduction, we have by transfer theorem,

$$H_{\text{et},c}^3(\mathcal{A}_{11}^{\text{lev}}, \mathbb{Q}_{\ell}) \xrightarrow{\sim} H_{\text{et},c}^3(S_{\Gamma(11)}, \mathbb{Q}_{\ell})^G \hookrightarrow H_{\text{et},c}^3(S_{\Gamma(11)}, \mathbb{Q}_{\ell}).$$

It is easy to see that this map is compatible with the pull-pack  $\pi^*$ . Note that  $\pi^*$  is injective because  $\pi$  is a finite map. Hence  $H_{\text{et},!}^3(\mathcal{A}_{11}^{\text{lev}}, \mathbb{Q}_{\ell})$  is a  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -submodule of  $H_{\text{et},!}^3(S_{\Gamma(11)}, \mathbb{Q}_{\ell})$ . Since  $S_{\Gamma(11)}$  is smooth, by Corollaire (3.3.6) of [5], the cohomology  $H_{\text{et},!}^3(S_{\Gamma(11)}, \mathbb{Q}_{\ell})$  is pure of weight 3 and so is  $H_{\text{et},!}^3(\mathcal{A}_{11}^{\text{lev}}, \mathbb{Q}_{\ell})$ .

REMARK 1.4. Theorem 1.3-(i) does not make sense if we do not tell about  $\mathbb{Q}$ -Hodge structures or period Lattices in those cohomologies. The period lattice  $H^3(X, \mathbb{Z})$  is determined by Roulleau [34]. If we determine  $H_{\text{et},!}^3(\mathcal{A}_{11}^{\text{lev}}, \mathbb{Z})$  and show that it is quasi-isomorphic to  $H^3(X, \mathbb{Z})$  under the map of Theorem 1.3-(i), we will be able to prove that  $L(s, H_{\text{et}}^3(X_{\overline{\mathbb{L}}}, \mathbb{Q}_{\ell}))$  occurs in  $L(s, H_{\text{et},!}^3(\mathcal{A}_{11}^{\text{lev}}, \mathbb{Q}_{\ell}))$  for some number field  $L$ .

Let  $\Gamma_{\mathbb{A}}$  be an open compact subgroup of  $\text{GSp}_2(\mathbb{A})$  such that  $\Gamma := \Gamma_{\mathbb{A}} \cap \text{Sp}_2(\mathbb{Q})$  is an arithmetic subgroup of  $\text{Sp}_2(\mathbb{Q})$  (not necessary torsion free). Let us denote by  $S_{\Gamma} = \Gamma \backslash \mathbb{H}_2$  the corresponding Siegel modular threefold. Let  $\mathcal{L}_{a,b}$  be the local system over  $S_{\Gamma}$  associated to an irreducible representation  $L_{a,b}$  over  $\mathbb{C}$  of  $\text{Sp}_2(\mathbb{R})$  with dominant highest weight  $(a, b)$  with  $a \geq b \geq 0$  (we assume that  $L_{a,b}$  comes from an irreducible algebraic representation of  $\text{Sp}_2$ ). Take a Galois cover  $\pi : S_{\Gamma'} \longrightarrow S_{\Gamma}$  with the Galois group  $G$  so that  $S_{\Gamma'}$  is a fine moduli space. Then we can define the ( $\ell$ -adic) étale local system  $\mathcal{L}_{a,b}^{\text{et}}$  on  $S_{\Gamma'}$  associated to  $L_{a,b}$  by using the Hodge bundle on  $S_{\Gamma'}$  (cf. Section 1 of [44]). Let  $\text{Gr}_3^W H_{\text{beti}}^3(S_{\Gamma\mathbb{C}}, \mathcal{L}_{a,b})$  be the graded quotient of degree three of a mixed Hodge structure on  $H_{\text{beti}}^3(S_{\Gamma\mathbb{C}}, \mathcal{L}_{a,b})$ .

In what follows, we will discuss our results with known results or conjectures about irreducible cuspidal automorphic representation  $\Pi$  of  $\text{GSp}_2(\mathbb{A})$  which arises from a differential form of  $\text{Gr}_3^W H_{\text{beti}}^3(S_{\Gamma\mathbb{C}}, \mathcal{L}_{a,b})$ . Assume that  $\Pi = \otimes_v \Pi_v$  is weakly equivalent to a representation of multiplicity one. Let  $L_{\Pi}$  be the set of irreducible cuspidal automorphic representations  $\Pi' = \otimes_v \Pi'_v$  of  $\text{GSp}_2(\mathbb{A})$  such that every  $\Pi'_v$  belongs to the local  $L$ -packet of  $\Pi_v$  for each place  $v$  of  $\mathbb{Q}$ . See [6] and [29] for the definition of the local  $L$ -packet. By Theorem 2.1 of [25], if  $\Pi' \in L_{\Pi}$ , then  $\Pi'_{\infty}|_{\text{Sp}_2}$  or  $\overline{\Pi'_{\infty}}|_{\text{Sp}_2}$  is the holomorphic discrete series representation with Blattner parameter  $(a+b+3, a+b+3)$  or the non-holomorphic discrete series one with Blattner parameter  $(a+3, -b-1)$  which is generic. For a  $\Pi' \in L_{\Pi}$ , let

$$c_{\Gamma}(\Pi') = \dim_{\mathbb{C}}\{f \in \Pi' \mid f \text{ is right } \Gamma_{\mathbb{A}}\text{-invariant}\}.$$

Let  $c_{\Gamma}^H(L_{\Pi})$  (resp.  $c_{\Gamma}^W(L_{\Pi})$ ) be the sum of  $c_{\Gamma}(\Pi')$  over  $\Pi' \in L_{\Pi}$  such that  $\Pi'_{\infty}|_{\text{Sp}_2}$  is the holomorphic (resp. non-holomorphic) discrete series representation with Blattner parameter  $(a+b+3, a+b+3)$  (resp.  $(a+3, -b-1)$ ). First of all, if  $\Pi$  is neither a weak endoscopic lift nor a CAP representation, then by Proposition 1.5 of Weissauer [44], for any  $\Pi' \in L_{\Pi}$  there exists  $\Pi''$  such that  $\Pi'_p \simeq \Pi''_p$  for all nonarchimedean  $p$  but  $\Pi'_{\infty} \not\simeq \Pi''_{\infty}$  and  $\Pi'_{\infty} \not\simeq \overline{\Pi''_{\infty}}$ . Therefore,  $c_{\Gamma}^H(L_{\Pi}) = c_{\Gamma}^W(L_{\Pi})$  and by THÉORÈME

7.5 of Laumon [18] and Theorem III of [44],

$$L_\Sigma(s, \Pi)^{c_\Gamma^H(L_\Pi)} | L_\Sigma(s, \text{Gr}_3^W H_{\text{et}}^3(S_{\Gamma, \overline{\mathbb{Q}}}, \pi_* \mathcal{L}_{a,b}^{\text{et}}))$$

where  $\Sigma$  is the set of finite places  $v$  of which  $\Gamma_v \not\cong \text{Sp}_2(\mathbb{Q}_v)$  and we set  $L_\Sigma(s, \Pi) := \prod_{v \notin \Sigma} L(s, \Pi_v)$ . However their results of [18] and [44] do not tell us anything about the contribution of a weak endoscopic lift or a CAP representation to the middle cohomology of any Siegel modular threefold. On the other hand we have a conjectural discription of the contribution of a weak endoscopic lift  $\Pi$  (or also a CAP representation) to the (total)  $\ell$ -adic cohomology of  $S_\Gamma$  (Section 6 of [41]). By Howe, Piatetski-Shapiro [12], there exists an endoscopic lift  $\Pi$  for a given pair  $(\pi_1, \pi_2)$  where  $\Pi$  is given by the  $\theta$ -lift from  $\text{GSO}_{2,2}$  and globally generic. Therefore  $c_\Gamma^W(L_\Pi) \neq 0$  for a sufficiently small  $\Gamma$ . We should note that

- if  $\Pi' \in L_\Pi$  and  $\Pi_\infty \not\cong \Pi'_\infty$ , then  $\Pi_p \not\cong \Pi'_p$  for some nonarchimedean  $p$ .
- it may be happen  $c_\Gamma^H(L_\Pi) = 0$  for any  $\Gamma$  (see section 5 for the explanation).

By Arthur's conjecture (cf. Section 6 of [41]) and  $p$ -adic Hodge theory, we guess

$$L_\Sigma(s, \pi_2)^{c_\Gamma^H(L_\Pi)} L_\Sigma(s, \pi_1)^{c_\Gamma^W(L_\Pi)} | L_\Sigma(s, \text{Gr}_3^W H_{\text{et}}^3(S_{\Gamma, \overline{\mathbb{Q}}}, \pi_* \mathcal{L}_{a,b}^{\text{et}})),$$

where  $\pi_1$  (resp.  $\pi_2$ ) is chosen so that  $\pi_{1,\infty}|_{\text{SL}_2}$  (resp.  $\pi_{2,\infty}|_{\text{SL}_2}$ ) is the discrete series representation of lowest weight  $a - b + 2$  (resp.  $a + b + 4$ ). In the following specific case, we would like to give a conjecture. Let  $S$  be a Siegel threefold defined over  $\mathbb{Q}$  with a Hecke correspondence  $\gamma \subset S \times S$  which is also defined over  $\mathbb{Q}$ . Assume

$$h^{3,0}(\gamma^* \text{Gr}_3^W H_{\text{betti}}^3(S_{\mathbb{C}}, \pi_* \mathcal{L}_{a,b})) = 0, \quad h^{2,1}(\gamma^* \text{Gr}_3^W H_{\text{betti}}^3(S_{\mathbb{C}}, \pi_* \mathcal{L}_{a,b})) = 1. \quad (1.2)$$

Let  $\Pi$  be the irreducible cuspidal automorphic representation attached to a unique generator of  $H^{2,1}(\gamma^* \text{Gr}_3^W H_{\text{betti}}^3(S_{\mathbb{C}}, \pi_* \mathcal{L}_{a,b}))$ . By Theorem 2.1 of [25],  $\Pi_\infty|_{\text{Sp}_2(\mathbb{R})}$  is the non-holomorphic discrete series representation with Blattner parameter  $(a + 3, -b - 1)$  which is generic. By Theorem III and Proposition 1.5 of [44],  $\Pi$  is a CAP representaiton or a weak endoscopic lift of a pair  $(\pi_1, \pi_2)$ . But, by section 4 of Schmidt [37], if a Saito-Kurokawa representation (a CAP representation associated to a Siegel parabolically induced representation) is not holomorphic, then its archimedean component is non-tempered and hence it is not a discrete series representation. By Soudry [38], every CAP representation associated to a Klingen or Borel parabolically induced representation is given by a  $\theta$ -lift from  $\text{GO}(L_{\mathbb{A}})$ , where  $L$  is a quadratic extension of  $\mathbb{Q}$ . It is not hard to show that the archimedean component of the  $\theta$ -lift is not generic. Hence,  $\Pi$  is a weak endoscopic lift. Then our conjecture is:

**CONJECTURE 1.5.** *Keep the notations as above. The following equality of  $L$ -functions holds up to finitely many local factors:*

$$L(s, \gamma^* \text{Gr}_3^W H_{\text{et}}^3(S_{\overline{\mathbb{Q}}}, \pi_* \mathcal{L}_{a,b}^{\text{et}})) = L(s - b - 1, f)$$

where  $f$  is the elliptic cusp form of weight  $a - b + 2$  associated to  $\pi_1$  (of the lower weight).

We should explain where the missing contribution of  $\pi_g \otimes \chi^i$  is gone in our case. Recall the Galois covering  $S_{\Gamma(11)}$  of  $\mathcal{A}_{11}^{\text{lev}}$ . We can construct a holomorphic weak endoscopic lift  $\Pi'_i = \Theta((\pi_f \otimes \chi^i) \boxtimes (\pi_g \otimes \chi^i))$  associated to a pair  $(\pi_f \otimes \chi^i, \pi_g \otimes \chi^i)$  such that  $\Pi'_i$  has a right  $\Gamma(11)_{\mathbb{A}}$ -invariant vector which contributes to  $(3, 0)$ -part of  $\text{Gr}_3^W H^3(S_{\Gamma(11)}, \mathbb{C})$ , hence  $|L_{\Pi'}| = 2$ . The construction of  $\Pi'_i$  is given by the first author [27]. If we expect Arthur's conjecture (cf. Section 6 of [41]),  $\Pi'_i$  should contribute to the  $\ell$ -adic cohomology of  $S_{\Gamma(11)}$  of the middle degree.

**REMARK 1.6.** (i) *The results of this paper can be viewed as a warning that the good behaviour predicted if  $\Gamma$  is torsion free may not occur in some case like  $\text{K}(11)^{\text{lev}}$ , due to the specific geometry of the corresponding Siegel threefold  $\mathcal{A}_{11}^{\text{lev}}$ . However we know fortunately that  $\mathcal{A}_{11}^{\text{lev}}$  is unirational. So*

we can discuss Theorem 1.3-(ii). In general, it seems to be hard to do by using (purely) geometric arguments.

(ii) It is irrelevant to conclude that the smallness of  $\Gamma$  causes this phenomena as above. In fact, if  $\Gamma$  is inadmissible (e.g.  $\Gamma = K(N)$  or  $K(N)^{\text{lev}}$  for any  $N$ ), then there are no contribution of holomorphic weak endoscopic lifts (hence Yoshida lifts) to  $(3,0)$ -part of  $\text{Gr}_3^W H^3(S_\Gamma, \mathbb{C})$ . We will discuss this in Section 5.

(iii) Let  $g$  be the elliptic cusp form of weight  $a+b+4$  associated to  $\pi_2$ . Since the Frobenius eigenvalues at  $p \neq \ell$  on  $\gamma^* \text{Gr}_3^W H_{\text{et}}^3(S_{\overline{\mathbb{Q}}}, \pi_* \mathcal{L}_{a,b}^{\text{et}})$  are of form  $p^{b+1} \alpha$  for some  $\alpha \in \mathbb{Z}_\ell$ , we see that the  $L$ -function of  $g$  never contribute to  $\gamma^* \text{Gr}_3^W H_{\text{et}}^3(S_{\overline{\mathbb{Q}}}, \pi_* \mathcal{L}_{a,b}^{\text{et}})$ .

The paper is organized as follows. In section 2, we determine the  $L$ -function of Klein's cubic hypersurface by using theory of Fano threefolds and motives. So we will freely use the terminology in [19] (see also [23] for a modern article). In section 3, we construct non-holomorphic differential forms  $F_i$  on  $\mathcal{A}_{11}^{\text{lev}}$ . In section 4, we give a proof of Theorem 1.3 and discuss a related topic in Section 5.

## 2. Klein's cubic threefold and its $L$ -function.

In this section, we compute  $L$ -function of Klein's cubic threefold  $X$  defined by  $x_0^2 x_1 + x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_4 + x_4^2 x_0 = 0$  in the projective space  $\mathbb{P}^4$  where we fix coordinates  $[x_0 : x_1 : x_2 : x_3 : x_4]$ . We consider  $X$  as a variety defined over  $\mathbb{Q}$ .

PROPOSITION 2.1. *The variety  $X$  has a good reduction outside 11.*

*Proof.* We can easily check this by direct computation, but we give another proof. Let  $F_{11} : \sum_{i=0}^4 y_i^{11} = 0$  be the Fermat hypersurface of degree 11 in  $\mathbb{P}_{\mathbb{Q}}^4$ . Then we have the generically finite, surjective morphism  $F_{11} \rightarrow X, (y_i)_i \mapsto (x_i)_i = (y_i^4 y_{i+1}^2 y_{i+2}^3 y_{i+3}^8)_{i \in \mathbb{Z}/5\mathbb{Z}}$  which is defined over  $\mathbb{Z}$ . Clearly  $F_{11}$  and the indeterminacy of the map as above have good reduction outside 11, hence so is  $X$ .

Let  $H_{\text{dR}}^*(Y)$  be the algebraic de Rham cohomology of a variety  $Y$  (may be affine or singular) over a field  $K$  which is defined to be the hypercohomology group of de Rham complex

$$\Omega_Y := [0 \rightarrow \mathcal{O}_Y \xrightarrow{d} \Omega_Y^1 \xrightarrow{d} \Omega_Y^2 \rightarrow \cdots]$$

(see [10]). It is a  $K$ -vector space by definition. If  $Y$  is affine, this coincides with the cohomology of the global sections. In general, it is hard to compute the algebraic de Rham cohomology. However, now  $X$  is a hypersurface of projective space. So we can apply Griffiths-Dwork's results to compute an explicit generators of  $H_{\text{dR}}^3(X)$ .

We now explain this. Put  $S := x_0^2 x_1 + x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_4 + x_4^2 x_0$  and denote by  $R = \mathbb{Q}[x_0, x_1, x_2, x_3, x_4]$  the polynomial ring over  $\mathbb{Q}$  with five variables and  $R_d$  the set of all homogeneous polynomial of degree  $d \in \mathbb{Z}_{\geq 0}$ . Consider  $U := \mathbb{P}^4 \setminus X$ . Then  $U$  is an affine variety over  $\mathbb{Q}$  and it has coordinate ring  $\Gamma(U, \mathcal{O}_U)$  which consists of the homogenous elements of degree 0 in  $R[\frac{1}{S}]$ .

THEOREM 2.2. ([8]) *There exists an isomorphism  $H_{\text{dR}}^3(X) \simeq H_{\text{dR}}^4(U)$  as  $\mathbb{Q}$ -vector spaces. Furthermore, this isomorphism commutes with the action of  $\text{Aut}(X) \cap \text{Aut}(\mathbb{P}^4)$ .*

*Proof.* This follows from the excision theorem and the later claim follows from the functoriality of cohomology.

So we have only to compute  $H_{\text{dR}}^4(U)$  instead of  $H_{\text{dR}}^3(X)$ . Put  $\Omega = \sum_{i=0}^4 x_i dx_0 \wedge \cdots \wedge \hat{dx}_i \wedge \cdots \wedge dx_4$ .



**THEOREM 2.3.** *The cohomology  $H_{\text{dR}}^4(U)$  consists of  $\frac{x_i \Omega}{S^2}$ ,  $\frac{\partial_i S \Omega}{S^3}$  and  $\frac{x_i \Omega}{S^2}$  gives the Hodge filtration  $\text{Fil}^2 H_{\text{dR}}^3(X)$ .*

*Proof.* Since  $U$  is affine,  $H_{\text{dR}}^4(U) = \text{Ker}(d : \Omega_U^4 \rightarrow \Omega_U^5 = 0) / \text{Im}(d : \Omega_U^3 \rightarrow \Omega_U^4)$ . Furthermore right hand side can be written as

$$\left\{ \frac{A \Omega}{S^i} \mid A \in R_{3i-5}, i = 2, 3, \dots \right\} / \left\{ \partial_j \left( \frac{A}{S^i} \right) \Omega \mid j = 0, 1, 2, 3, 4, A \in R_{3i-4}, i = 2, 3, \dots \right\}.$$

By direct computation, we can find generators as in the claim. We know a priori that  $H_{\text{dR}}^3(X)$  (resp.  $\text{Fil}^2 H_{\text{dR}}^3(X)$ ) is of dimension 10 (resp. 5). So this would help us from abstract computations.

By [8],  $\text{Fil}^2 H_{\text{dR}}^3(X)$  corresponds to the image of  $\left\{ \frac{A \Omega}{S^{i+1}} \mid A \in R_{3i-2}, i = 1, 2, \dots \right\}$  in  $H_{\text{dR}}^4(U)$ . Then we have the last claim with computations above.

Let  $\zeta_5$  be a primitive 5-th root of unity.

**PROPOSITION 2.4.** *Let  $\alpha$  be the automorphism on  $X$  defined by  $[x_0 : x_1 : x_2 : x_3 : x_4] \mapsto [x_1 : x_2 : x_3 : x_4 : x_0]$  ( $\alpha$  is of order five) and  $\alpha^*$  be the corresponding linear map on  $H_{\text{dR}}^3(X)$ . Then  $H_{\text{dR}}^3(X) \otimes \mathbb{Q}(\zeta_5)$  decomposes as  $\bigoplus_{i=0}^4 W(\zeta_5^i)$  where  $\alpha^*$  acts on 2-dimensional space  $W(\zeta_5^i)$  over  $\mathbb{Q}(\zeta_5)$  as multiplication  $\zeta_5^i$ . Furthermore, this decomposition preserves Hodge filtration.*

*Proof.* By Theorem 2.3, it is easy to see that

$$v_j = \sum_{i=0}^4 \zeta_5^{j(i+1)} \frac{x_i \Omega}{S^2}, \quad \omega_j = \sum_{i=0}^4 \zeta_5^{j(i+1)} \frac{\partial_i S \Omega}{S^3}, \quad j = 0, 1, 2, 3, 4$$

is a generator of  $W(\zeta_5^i)$ . Since  $\text{Fil}^2 H_{\text{dR}}^3(X) \cap W(\zeta_5^i) = \langle v_i \rangle$ , this decomposition preserves Hodge filtration.

**THEOREM 2.5.** *Let  $f$  be the elliptic modular form of weight 2 with complex multiplication by the ring of integers of  $\mathbb{Q}(\sqrt{-11})$ . Then we have  $L(s, H_{\text{et}}^3(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)) = \prod_{i=0}^4 L(s-1, f \otimes \chi^i)$  including local  $L$ -factor at 11 where  $\chi : (\mathbb{Z}/11\mathbb{Z})^* \rightarrow \mathbb{C}^\times, \bar{2} \mapsto \zeta_5$  is a primitive character of order 5 with conductor 11. In particular,  $L(s, H_{\text{et}}^3(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell))$  is independent to a choice of  $\ell$ .*

*Proof.* Let  $E$  be the elliptic curve over  $\mathbb{Q}$  which corresponds to  $f$ . Let  $S$  be the Hilbert scheme of lines of  $X$  which is a smooth surface over  $\mathbb{Q}$ . Then by the general theory of Fano threefold (cf. [3], [19]), the Grothendieck motive  $M := h^3(X)$  associated to  $X$  over  $\mathbb{Q}$  coincides with the motives  $h^1(A)(1)$  over  $\mathbb{Q}$  associated to the Albanese variety  $A$  of  $S$  where “(1)” means the twist by Lefschetz motive. Note that  $X$  has the Chow-Künneth decomposition by [19]. The motive  $h^3(X)$  in fact exists in the category of Grothendieck motives. So we have

$$L(s, H_{\text{et}}^3(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)) = L(s, M) = L(s, h^1(A)(1)) = L(s-1, H_{\text{et}}^1(A_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)).$$

It is known by Theorem (46.22) in [1] that as abelian varieties,  $A$  is isomorphic to  $E^5$  over  $\mathbb{C}$ . Recall  $\alpha$  is an automorphism of order 5 defined in Proposition 2.4. By functoriality,  $\alpha$  is identified with an element in  $\text{End}_{\mathbb{C}}(A) \otimes \mathbb{Q} = \text{End}_{\mathbb{C}}(E^5) \otimes \mathbb{Q} = M_5(\text{End}_{\mathbb{C}}(E) \otimes \mathbb{Q}) = M_5(K)$ . Since  $\alpha$  is of order five, this must be a permutation of order five such as

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \in M_5(\mathbb{Q}(\sqrt{-11})).$$

Therefore, we see that  $\alpha - 1 \in \text{End}_{\mathbb{Q}}(A) \otimes \mathbb{Q}$  is a non-trivial zero divisor and then  $B := A/(\alpha - 1)A$  is an one dimensional abelian variety over  $\mathbb{Q}$ . In fact, by Proposition 2.4, the the filtration of de Rham realization  $H_{\text{dR}}^3(X)_{\mathbb{Q}}$  of  $M$  has only one dimensional eigenspace (as a  $\mathbb{Q}$ -vector space) for eigenvalue 1 of  $\alpha^*$ . Since  $B$  has CM by  $\mathcal{O} = \mathcal{O}_K$ ,  $\sharp \mathcal{O}^\times = 2$  and  $B$  has conductor a power of 11 by Proposition 2.1, then  $B$  is  $E$  or the twist of  $E$  by  $K$ . The latter case becomes  $E$  again.

Consider the quotient abelian variety  $B' = A/B$ . By direct computation, we have the local  $L$ -factor at 3 of  $X$  and hence of  $B \times B'$  up to Tate twists:

$$L_3(s, H_{\text{et}}^3(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)) = (1 + 3x + 27x^2) \times (1 - 3x - 18x^2 + 135x^3 + 81x^4 + 3645x^5 - 13122x^6 - 59049x^7 + 531441x^8),$$

where  $x = 3^{-s}$ . Since the second factor of the right hand side is irreducible as a polynomial over  $\mathbb{Q}$ , we see that  $B'$  is a  $\mathbb{Q}$ -simple abelian variety. We denote by  $\text{End}_{\mathbb{Q}}(B')$  the ring of endomorphisms defined over  $\mathbb{Q}$  of  $B'$ . This is a  $\mathbb{Z}$ -algebra. Consider the composite of the following homomorphisms:

$$L = \mathbb{Q}(\zeta_5) \xrightarrow{\zeta_5 \mapsto \alpha} \text{End}_{\mathbb{Q}}(A) \otimes_{\mathbb{Z}} \mathbb{Q} = \text{End}_{\mathbb{Q}}(B) \otimes_{\mathbb{Z}} \mathbb{Q} \times \text{End}_{\mathbb{Q}}(B') \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \text{End}_{\mathbb{Q}}(B') \otimes_{\mathbb{Z}} \mathbb{Q},$$

where the second homomorphism is the natural projection. Since  $\text{End}_{\mathbb{Q}}(B) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$ , this map gives an embedding  $L \hookrightarrow \text{End}_{\mathbb{Q}}(B') \otimes_{\mathbb{Z}} \mathbb{Q}$ . Then  $B'$  is an abelian variety of  $\text{GL}_2$ -type in the sense of Ribet [31] and Theorem 4.4 loc.cit with Serre's conjecture which is now a theorem by [16],  $B'$  is isogenous to the Shimura's abelian variety  $A_h$  for some elliptic modular form  $h$  (see Theorem 7.14 of [39] for  $A_h$ ).

On the other hand,  $B'_{\mathbb{Q}}$  is isogenous to  $E^4$  over  $\overline{\mathbb{Q}}$  and  $E$  is corresponding to the CM modular forms  $f \in S_2(\Gamma_0(11^2))$ . Note that  $B'$  has good reduction outside 11 by Proposition 2.1. Then by Theorem 1.2 in [7], we may assume that  $B'_{\mathbb{Q}}$  is isogenous to  $E^4$  over a number field  $K$  included in  $\mathbb{Q}(\mu_{11^\infty})$ . Since  $\text{Gal}(K/\mathbb{Q})$  is abelian, by taking Weil restriction, we must have  $h = f \otimes \psi$  for some primitive character  $\psi$ . Note that  $L = \mathbb{Q}(a_n(h) | n \geq 1)$  by Theorem 7.14 of [39]. So we may have  $\psi = \chi$ . Hence we have

$$L(s, H_{\text{et}}^3(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)) = L(s - 1, H_{\text{et}}^1(A_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)) = \prod_{i=0}^4 L(s - 1, H_{\text{et}}^1(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \otimes \chi^i) = \prod_{i=0}^4 L(s - 1, f \otimes \chi^i).$$

For the local  $L$ -factor at 11, the last equality follows from the local-global compatibility of automorphic  $L$ -function (cf. [4],[35]). In particular, the LHS is independent to  $\ell$ .

### 3. construction of right $K(11)_{\mathbb{A}}^{\text{lev}}$ -invariant cusp forms.

For  $k = \mathbb{Q}, \mathbb{Q}_v$ , or  $\mathbb{A}$ , let

$$\text{GSp}_n(k) = \left\{ g \in \text{GL}_{2n}(k) \mid {}^t g \begin{bmatrix} 0_n & -1_n \\ 1_n & 0_n \end{bmatrix} g = c(g) \begin{bmatrix} 0_n & -1_n \\ 1_n & 0_n \end{bmatrix}, c(g) \in k^\times \right\}$$

where  $c(g)$  is the similitude norm of  $g$ . Note that  $\text{GSp}_1(k) \simeq \text{GL}_2(k)$ . For a representation  $\tau$  of  $\text{GSp}_n(k)$  and a quasi-character  $\lambda$ , we will denote by  $\lambda\tau$  the representation sending  $g$  to  $\lambda(c(g))\tau(g)$ . For a positive integer  $N$ , the paramodular groups  $K(N)$  and  $K(N)^{\text{lev}}$  with a canonical level structure

are defined by

$$K(N) = \left\{ \begin{bmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N\mathbb{Z} \\ N\mathbb{Z} & \mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N\mathbb{Z} \\ \mathbb{Z} & N^{-1}\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{bmatrix} \right\} \cap \mathrm{Sp}_2(\mathbb{Q}),$$

$$K(N)^{\mathrm{lev}} = \left\{ \begin{bmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N\mathbb{Z} \\ N\mathbb{Z} & 1 + N\mathbb{Z} & N\mathbb{Z} & N^2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & 1 + N\mathbb{Z} \end{bmatrix} \right\} \cap \mathrm{Sp}_2(\mathbb{Q}).$$

For a nonarchimedean place  $v$  of  $\mathbb{Q}$ , let  $K(N)_v$  and  $K(N)_v^{\mathrm{lev}}$  be the  $v$ -adic completions of  $K(N)$  and  $K(N)^{\mathrm{lev}}$ . Let  $K(N)_{\mathbb{A}} = \prod_{v < \infty} K(N)_v$  and  $K(N)_{\mathbb{A}}^{\mathrm{lev}} = \prod_{v < \infty} K(N)_v^{\mathrm{lev}}$ . In this section, put  $p = 11$ . Let  $\chi = \otimes \chi_v$  be a primitive character of  $\mathbb{Q}^\times \backslash \mathbb{A}^\times$  of order 5 with conductor  $p$ . Let  $f \in S_2(\Gamma_0(11^2))$  be the elliptic CM modular form. Let  $\mu$  be the größencharacter of  $K = \mathbb{Q}(\sqrt{-11})$  associated to  $f$ . Let

$$\nu = \frac{\mu}{|\mu|}$$

and  $\pi_1 = \pi(\nu)$  be the irreducible cuspidal automorphic representation of  $\mathrm{PGL}_2(\mathbb{A})$  associated to  $\nu$ . We will construct right  $K(p)_{\mathbb{A}}^{\mathrm{lev}}$ -invariant non-holomorphic automorphic forms corresponding to non-holomorphic differential forms on  $H^{2,1}(\mathrm{Gr}_3^W H_{\mathrm{betti}}^3(\mathcal{A}_{1,11}^{\mathrm{lev}}, \mathbb{C}))$ . Let  $\Pi$  be an irreducible cuspidal automorphic representation of  $\mathrm{GSp}_2(\mathbb{A})$  which arises from a non-holomorphic differential form on  $H^{2,1}(\mathrm{Gr}_3^W H_{\mathrm{betti}}^3(\mathcal{A}_{1,11}^{\mathrm{lev}}, \mathbb{C}))$ . By the similar argument done before Conjeceture 1.5 and Tilouine's conjectural panorama of [41], we guess that  $\Pi$  is a weak endoscopic lift of a pair  $(\chi^i \pi_1, \chi^i \pi_2)$  (see eq.(2) at p. 505 of [26] for the case of  $a = b = 0$ ). Here  $\pi_2$  is some irreducible cuspidal automorphic representation of  $\mathrm{PGL}_2(\mathbb{A})$  with  $\pi_{2,\infty}$  being the discrete series representation of lowest weight 4. We can assume that the central character of  $\Pi$  is trivial, since a weak endoscoic lift of  $(\chi^i \pi_1, \chi^i \pi_2)$  is a  $\chi^i$ -twist of that of  $(\pi_1, \pi_2)$ . By Roberts [32], every weak endoscopic lift is given by a global  $\theta$ -lift from  $\mathrm{GSO}_B(\mathbb{A})$  for some quaternion algebra  $B$  defined over  $\mathbb{Q}$ . We recall some fundamental results on the  $\theta$ -lifts. Define the action  $\rho$  of  $B^\times \times B^\times$  on  $B$  by  $\rho(h_1, h_2)x = h_1^{-1}xh_2$ , which yields an isomorphism

$$i_\rho : \mathrm{GSO}(B) \simeq B^\times \times B^\times / \Delta(\mathbb{Q}^\times)$$

where  $\Delta$  indicates the diagonal embedding. Let  $\pi_i^B$  be the Jacquet-Langlands transfer of  $\pi_i$  to  $B(\mathbb{A})^\times$  if it exists. By  $i_\rho$ , a pair of automorphic representations  $(\pi_1^B, \pi_2^B)$  of  $B(\mathbb{A})^\times$  is identified with an automorphic representation of  $\mathrm{GSO}(B(\mathbb{A}))$ . We denote by  $\Theta(\pi_1^B \boxtimes \pi_2^B)$  the global  $\theta$ -lift and by  $\theta(\pi_{1,v}^B \boxtimes \pi_{2,v}^B)$  the local  $\theta$ -lift, which is the  $v$ -component of  $\Theta(\pi_1^B \boxtimes \pi_2^B)$ . If  $B = M_2(\mathbb{Q})$ , we write  $\Theta(\pi_1 \boxtimes \pi_2)$  as  $\Theta(\pi_1^B \boxtimes \pi_2^B)$  and  $\theta(\pi_{1,v} \boxtimes \pi_{2,v})$  as  $\theta(\pi_{1,v}^B \boxtimes \pi_{2,v}^B)$ , briefly. Then,

$$\theta(\pi_{1,v}^B \boxtimes \pi_{2,v}^B) \simeq \theta(\pi_{1,v} \boxtimes \pi_{2,v}) \quad (\Longleftrightarrow) \quad B_v \simeq M_2(\mathbb{Q}_v).$$

First, we should find a candidate of  $B$  and  $\pi_2$  such that  $\Pi = \Theta(\pi_1^B \boxtimes \pi_2^B)$ . If  $B_\infty$  is not split, then  $\theta(\pi_{1,\infty}^B \boxtimes \pi_{2,\infty}^B)$  is holomorphic. Hence,  $B_\infty$  should be split. At a nonarchimedean place  $v$ ,  $\theta(\pi_{1,v}^B \boxtimes \pi_{2,v}^B)$  is unramified, if and only if  $B_v$  is split and both of  $\pi_{1,v}, \pi_{2,v}$  are unramified. Hence  $\pi_{2,v}$  is unramified and  $B_v$  is split for  $v \neq p$  since  $\Pi_v$  is unramified. Therefore, we conclude that  $B = M_2(\mathbb{Q})$  by the Hasse principle, and  $\pi_2$  should have a  $p$ -power conductor. Moreover, we know that the local  $\theta$ -lift  $\theta(\pi_{1,p} \boxtimes \pi_{1,p})$  is a constituent (denoted by  $\tau(S, \pi_{1,p})$  in [33]) of the local Klingen parabolically induced representation  $1 \rtimes \pi_{1,p}$ , and have a right  $\Gamma_p$ -invariant vector for a relatively large compact subgroup  $\Gamma_p \subset \mathrm{Sp}_2(\mathbb{Q}_p)$ . Further,  $\pi(\nu^3)_p \simeq \pi(\nu)_p$  by the following lemma. Therefore, we choose  $\pi_2 = \pi(\nu^3)$ .

LEMMA 3.1. *The  $p$ -component  $\pi_{1,p} = \pi(\nu)_p$  of  $\pi_1$  is supercuspidal and  $\pi(\nu)_p = \pi(\nu^3)_p$ .*



*Proof.* Let  $\mathfrak{p} = \sqrt{-11}$ , the place of  $K$  lying over  $p$ . If we assume that  $\pi(\nu)_p$  is not supercuspidal, then, by Theorem 4.6. (iii) of [14], we can write

$$\nu_{\mathfrak{p}} = \chi_p \circ N_{K_{\mathfrak{p}}/\mathbb{Q}_p} \quad (3.1)$$

by some character  $\chi_p$  on  $\mathbb{Q}_p^\times$ . Observing that  $[\mathbb{Z}_p^\times : N_{K_{\mathfrak{p}}/\mathbb{Q}_p}(\mathfrak{o}_{\mathfrak{p}}^\times)] = 2$  where  $\mathfrak{o}_{\mathfrak{p}}$  is the ring of integers of  $K_{\mathfrak{p}}$  and

$$(\mathbb{Z}_p/p^n\mathbb{Z}_p)^\times \cong \mathbb{Z}/p^{n-1}\mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z},$$

we conclude that  $(\chi_p|_{\mathbb{Z}_p^\times})^2 = 1$ . Take a rational prime  $l$  which is inert in  $K$  so that  $\nu_w(l) = 1$  at every  $w \neq \mathfrak{p}, \infty$ . Note that there are infinitely many such primes by Dirichlet's arithmetic progression theorem. Then,  $\nu_{\mathfrak{p}}(l) = \chi_p(N_{K_{\mathfrak{p}}/\mathbb{Q}_p}(l)) = \chi_p(l^2) = 1$ . Therefore,  $L(s, \nu)_l = (1 - l^{-2s})^{-1}$ , which conflicts to the form of the  $l$ -factor of  $L(s + \frac{1}{2}, E)$ . Hence  $\pi(\nu)_p$  is supercuspidal.

By the theory of complex multiplication,  $\mu = \otimes_w \mu_w$  takes the values in  $K^\times$ . Therefore  $\nu_{\mathfrak{p}}|_{\mathfrak{o}_{\mathfrak{p}}^\times}$  is  $\pm 1$ -valued since the units group of  $K^\times$  is  $\{\pm 1\}$ . Thus  $\nu_{\mathfrak{p}}|_{\mathfrak{o}_{\mathfrak{p}}^\times} = \nu_{\mathfrak{p}}^3|_{\mathfrak{o}_{\mathfrak{p}}^\times}$ , and  $\nu_{\mathfrak{p}}(\varpi_{\mathfrak{p}}) = \pm 1$  or  $\pm\sqrt{-1}$  for a uniformizer  $\varpi_{\mathfrak{p}}$  of  $K_{\mathfrak{p}}$ . Hence  $\nu_{\mathfrak{p}}$  coincides with  $\nu_{\mathfrak{p}}^3$  or  $\overline{\nu}_{\mathfrak{p}}^3$ . In any cases,

$$\pi_{1,p} = \pi(\nu_{\mathfrak{p}}) = \pi(\nu_{\mathfrak{p}}^3) = \pi(\overline{\nu}_{\mathfrak{p}}^3) = \pi_{2,p}.$$

Here note that central characters of  $\pi_1, \pi_2$  are trivial. This completes the proof.

**REMARK 3.1.** By the above argument and Lemma 3.1, the  $L$ -packet of the weak endoscopic lift associated to  $(\pi(\nu), \pi(\nu^3))$  is  $\{\Theta(\pi(\nu) \boxtimes \pi(\nu^3)), \Theta(\pi(\nu)^D \boxtimes \pi(\nu^3)^D)\}$ , where  $D$  is the definite quaternion algebra which is not split at only  $\infty$  and  $p$ .

We should note that if  $\Pi$  is a weak endoscopic lift of  $(\chi^i \pi_1, \chi^i \pi_2)$ , the spinor  $L$ -function of  $\Pi$  has the factor  $L(s, \chi^i \pi_2)$  which does not appear in the  $L$ -function of  $X$  (see Theorem 1.3(ii)).

Next, we are going to construct a right  $K(p)_{\mathbb{A}}^{\text{lev}}$ -invariant Whittaker function of  $\Theta(\pi_1 \boxtimes \pi_2)$ . Let

$$\begin{aligned} H(k) &= \text{GL}_2(k)^2 \\ H^1(k) &= \{(h_1, h_2) \in H(k) \mid \det(h_1) = \det(h_2)\} \end{aligned} \quad (3.2)$$

for  $k = \mathbb{Q}, \mathbb{Q}_v$  or  $\mathbb{A}$ . We will identify elements of  $H(k)$  with those of  $\text{GSO}_{\text{M}_2(k)} = \text{GSO}_{2,2}(k)$  via  $i_\rho$ . Let

$$e_1 = \begin{bmatrix} 0 & \frac{1}{p} \\ 0 & 0 \end{bmatrix}, \quad \alpha = \begin{bmatrix} \frac{1}{p} & 0 \\ 0 & -\frac{1}{p} \end{bmatrix} \in \text{M}_2(\mathbb{Q}).$$

Let  $Z_{(e_1, \alpha)}(k) \subset H^1(k)$  be the pointwise stabilizer subgroup of  $e_1, \alpha$ , which is isomorphic to

$$\left\{ \left( \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} \right) \mid x \in k \right\}.$$

We fix the standard additive character  $\psi = \otimes_v \psi_v$  on  $\mathbb{Q} \backslash \mathbb{A}$ . Let  $f_i$  be an automorphic form in  $\pi_i$ . Let  $r_v$  be the Weil representation of  $\text{Sp}_2(\mathbb{Q}_v) \times \text{O}_{2,2}(\mathbb{Q}_v)$  with respect to  $\psi_v$  on  $\mathcal{S}(\text{M}_2(\mathbb{Q}_v)^2)$ , the space of Schwartz-Bruhat functions of  $\text{M}_2(\mathbb{Q}_v)^2$ . Then an automorphic form  $\theta(\varphi, f_1 \boxtimes f_2)$  in  $\Theta(\pi_1 \boxtimes \pi_2)$  is

$$\theta(\varphi, f_1 \boxtimes f_2)(g) = \int_{H^1(\mathbb{Q}) \backslash H^1(\mathbb{A})} \sum_{x \in \text{M}_2(\mathbb{Q})^2} (\otimes_v r_v(g, h) \varphi_v(x)) f_1(h_1) f_2(h_2) dh_1 dh_2$$

where  $\varphi_v \in \mathcal{S}(\text{M}_2(\mathbb{Q}_v)^2)$  and  $dh_1, dh_2$  are Haar measures. Let  $W_i = \otimes_v W_{i,v}$  be the global Whittaker function with respect to  $\psi$  of  $f_i$ . Then, the  $v$ -component of the standard Whittaker function of  $\theta(\varphi, f_1 \boxtimes f_2)$  is

$$W_v(g) = \int_{Z_{(e_1, \alpha)}(\mathbb{Q}_v) \backslash H^1(\mathbb{Q}_v)} r_v(g, h) \varphi_v(e_1, \alpha) W_{1,v}(h_1) W_{2,v}(h_2) dh_1 dh_2. \quad (3.3)$$

It is easy to see  $W_\infty(1) \neq 0$  (see Remark 3.4 for an explicit  $\varphi_\infty$ ). Let  $\varphi_v = \text{Ch}(\text{M}_2(\mathbb{Z}_v)^2)$  for every nonarchimedean place  $v \neq p$ , where  $\text{Ch}$  indicates the characteristic function. Then, it is also easy to  $W_v(1) \neq 0$ , since both of  $\pi_{1,v}, \pi_{2,v}$  are unramified and we can take a right  $\text{GL}_2(\mathbb{Z}_v)$ -invariant  $W_{i,v}$ . Consequently, what we have to do is to choose  $\varphi_p$  and  $W_{i,p}$  suitably for the realization of a nontrivial right  $K(p)^{\text{lev}}$ -invariant  $W_p$ . Let

$$\beta = W_{1,p}^{\text{new}} = W_{2,p}^{\text{new}}$$

be the new vector of  $\pi_{1,p} \simeq \pi_{2,p}$ , which is right  $\Gamma_0(p^2)_p$ -invariant. We can assume  $\beta(1) = 1$ . Then it holds

$$\beta\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{cases} 1 & \text{if } a \in \mathbb{Z}_p^\times \\ 0 & \text{otherwise} \end{cases} \quad (3.4)$$

since  $\pi_{1,p}$  is supercuspidal by Lemma 3.1. We define

$$\varphi_p^{\text{lev}}(x_1, x_2) = \text{Ch}\left(\begin{bmatrix} \mathbb{Z}_p & p^{-1}\mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{bmatrix} \oplus \begin{bmatrix} p^{-1}\mathbb{Z}_p^\times & p^{-1}\mathbb{Z}_p \\ p\mathbb{Z}_p & p^{-1}\mathbb{Z}_p^\times \end{bmatrix}\right).$$

By using the properties of the Weil representation in p. 256 of [32], we can check that

$$r_p(u, (h_1, h_2))\varphi_p^{\text{lev}} = \varphi_p^{\text{lev}} \quad (3.5)$$

for  $u \in K(p)_p^{\text{lev}}$  and  $(h_1, h_2) \in ((\Gamma_0(p^2)_p \times \Gamma_0(p^2)_p) \cap H^1(\mathbb{Q}_p))$ . Now then, let us calculate  $W_p(1)$  using (3.3). Let

$$\begin{aligned} \tilde{\Gamma} &= \begin{bmatrix} p^2 & 0 \\ 0 & 1 \end{bmatrix} \text{GL}_2(\mathbb{Z}_p) \begin{bmatrix} p^2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \mathbb{Z}_p & p^{-2}\mathbb{Z}_p \\ p^2\mathbb{Z}_p & \mathbb{Z}_p \end{bmatrix} \cap \text{GL}_2(\mathbb{Q}_p) \simeq \text{GL}_2(\mathbb{Z}_p). \end{aligned}$$

As a complete system of representatives for  $\tilde{\Gamma}/\Gamma_0(p^2)_p$ , we can take

$$\left\{ \begin{bmatrix} 1 & j \\ 0 & 1 \end{bmatrix} \mid j \in p^{-2}\mathbb{Z}/\mathbb{Z} \right\} \sqcup \left\{ \begin{bmatrix} 0 & -p^{-1} \\ p & 0 \end{bmatrix} \begin{bmatrix} 1 & j \\ 0 & 1 \end{bmatrix} \mid j \in p^{-1}\mathbb{Z}/\mathbb{Z} \right\}.$$

Therefore, as a system of complete representatives of  $Z_{(e_1, \alpha)}(\mathbb{Q}_p) \backslash H^1(\mathbb{Q}_p) / (\Gamma_0(p^2)_p \times \Gamma_0(p^2)_p)$  we can take the following.

TYPE I):

$$\left( p^r \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p^m & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} p^n & 0 \\ 0 & 1 \end{bmatrix} \right)$$

with  $x \in \mathbb{Q}_p$ ,  $m + 2r = n$ .

TYPE II):

$$\left( p^r \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p^m & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ p^2 & 0 \end{bmatrix} \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} p^n & 0 \\ 0 & 1 \end{bmatrix} \right)$$

with  $x \in \mathbb{Q}_p$ ,  $2r + m + 2 = n$ , and  $s \in \{0, \frac{1}{p}, \dots, \frac{p-1}{p}\}$ .

TYPE III):

$$\left( p^r \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p^m & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} p^n & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ p^2 & 0 \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \right)$$

with  $x \in \mathbb{Q}_p$ ,  $m = n + 2 - 2r$  and  $t \in \{0, \frac{1}{p}, \dots, \frac{p-1}{p}\}$ .

TYPE IV):

$$\left( p^r \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p^m & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ p^2 & 0 \end{bmatrix} \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} p^n & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ p^2 & 0 \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \right)$$

with  $x \in \mathbb{Q}_p$ ,  $2r + m = n$ , and  $s, t \in \{0, \frac{1}{p}, \dots, \frac{p-1}{p}\}$ .

Let us see the contribution of each type of  $h = (h_1, h_2)$  in (3.3) to  $W_p(1)$ . We will write

$$\rho(h)((e_1, \alpha)) = \left( \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \right).$$

TYPE I). If an element  $h = (h_1, h_2)$  contributes to  $W_p(1)$ , at least,  $\rho(h)((e_1, \alpha)) \in \text{supp}(\varphi_p^{\text{lev}})$  and  $\beta(h_1)\beta(h_2) \neq 0$ . Therefore, by (3.4) and (3.5), we can assume

$$\left( \begin{bmatrix} 0 & p^{-1-m-r} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} p^{-1-m+n-r} & p^{-1-m-r}x \\ 0 & p^{-1-r} \end{bmatrix} \right) \in \begin{bmatrix} \mathbb{Z}_p & p^{-1}\mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{bmatrix} \oplus \begin{bmatrix} p^{-1}\mathbb{Z}_p^\times & p^{-1}\mathbb{Z}_p \\ p\mathbb{Z}_p & p^{-1}\mathbb{Z}_p^\times \end{bmatrix}$$

with  $m + 2r = n, m \geq 0$ . Observing  $a_2, d_2$  (resp.  $c_1$ ), we have  $r = 0, n = m$  (resp.  $m \leq 0$ ). Thus

$$\rho(h)((e_1, \alpha)) \in \text{supp}(\varphi_p^{\text{lev}}) \iff m = n = r = 0, x \in \mathbb{Z}_p.$$

Therefore, the total contribution of this type in (3.3) is

$$\text{vol}(\Gamma_0(p^2)_p \times \Gamma_0(p^2)_p) r_p(1, h) \varphi_p(e_1, \alpha) W_{1,p}(1) W_{2,p}(1) = \text{vol}(\Gamma_0(p^2)_p \times \Gamma_0(p^2)_p).$$

TYPE II). By (3.4) and (3.5), we can assume

$$\left( \begin{bmatrix} 0 & p^{-1-m-r}s \\ 0 & -p^{-1-m-r} \end{bmatrix}, \begin{bmatrix} p^{-1-m+n-r}s & p^{-3-m-r}(-p^m + p^2sx) \\ p^{-1-m+n-r} & -p^{-1-m-r}x \end{bmatrix} \right) \\ \in \begin{bmatrix} \mathbb{Z}_p & p^{-1}\mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{bmatrix} \oplus \begin{bmatrix} p^{-1}\mathbb{Z}_p^\times & p^{-1}\mathbb{Z}_p \\ p\mathbb{Z}_p & p^{-1}\mathbb{Z}_p^\times \end{bmatrix}$$

with  $2r + m + 2 = n \geq 0$  and  $s \in \{0, \frac{1}{p}, \dots, \frac{p-1}{p}\}$ . Observing  $a_2$ , we have  $s \neq 0$ . Observing  $b_1$  and  $a_2$ , we have  $n \leq 0$ . Thus  $n = 0$ . Observing  $c_2$ , we have  $-1 - m - r \geq 1$ . Observing  $b_2, d_2$ , we conclude that, if  $\rho(h)(e_1, \alpha) \in \text{supp}(\varphi_p^{\text{lev}})$ , then

$$\rho\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} h_1, h_2\right)(e_1, \alpha) \in \text{supp}(\varphi_p^{\text{lev}}) \quad (3.6)$$

for any  $x \in p^{-1}\mathbb{Z}_p$ . But, by the property

$$\beta\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} h_1\right) = \psi_p(x)\beta(h_1), \quad (3.7)$$

the contribution of this type is canceled.

TYPE III). By (3.4) and (3.5), we can assume

$$\left( \begin{bmatrix} p^{1-m-r} & p^{1-m-r}t \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} p^{1-m-r}x & p^{-1-m-r}(-p^n + p^2tx) \\ -p^{1-r} & -p^{1-r}t \end{bmatrix} \right) \\ \in \left( \begin{bmatrix} \mathbb{Z}_p & p^{-1}\mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{bmatrix} \oplus \begin{bmatrix} p^{-1}\mathbb{Z}_p^\times & p^{-1}\mathbb{Z}_p \\ p\mathbb{Z}_p & p^{-1}\mathbb{Z}_p^\times \end{bmatrix} \right)$$

with  $m = n + 2 - 2r \geq 0$  and  $t \in \{0, \frac{1}{p}, \dots, \frac{p-1}{p}\}$ . Observing  $c_2$ , we have  $r \leq 0$ . Thus,  $d_2 = -p^{1-r}t$  cannot belong to  $p^{-1}\mathbb{Z}_p^\times$ . This type does not contribute.

TYPE IV). By (3.4) and (3.5), we can assume

$$\left( \begin{bmatrix} p^{1-m-r}s & p^{1-m-r}st \\ -p^{1-m-r} & -p^{1-m-r}t \end{bmatrix}, \begin{bmatrix} p^{-1-m-r}(-p^m + p^2sx) & -p^{-1-m-r}(p^n s + p^m t - p^2 stx) \\ -p^{1-m-r}x & -p^{-1-m-r}(p^n - p^2 tx) \end{bmatrix} \right) \\ \in \left( \begin{bmatrix} \mathbb{Z}_p & p^{-1}\mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{bmatrix} \oplus \begin{bmatrix} p^{-1}\mathbb{Z}_p^\times & p^{-1}\mathbb{Z}_p \\ p\mathbb{Z}_p & p^{-1}\mathbb{Z}_p^\times \end{bmatrix} \right).$$

Here  $2r + m = n$  and  $s, t \in \{0, \frac{1}{p}, \dots, \frac{p-1}{p}\}$ . Observing  $c_1$ , we have  $-m - r \geq 0$ . If  $-m - r > 0$ , then the contribution is canceled by (3.6) and (3.7). Hence, we can assume  $m + r = 0, n = r$ . Then, since  $c_2 = -px \in p\mathbb{Z}_p$ , we have  $x \in \mathbb{Z}_p$ . Since  $d_2 = -p^{-1}(p^n - p^2tx) \in p^{-1}\mathbb{Z}_p$ , we have

$$p^{n-1} \in ptx + p^{-1}\mathbb{Z}_p = p^{-1}\mathbb{Z}_p.$$

Thus  $n \geq 0$ . Since  $a_2 = p^{-1}(-p^m + p^2sx) \in p^{-1}\mathbb{Z}_p$ , we have

$$p^{m-1} \in psx + p^{-1}\mathbb{Z}_p = p^{-1}\mathbb{Z}_p.$$

Thus  $m \geq 0$ , and  $n = r = -m \leq 0$ . Hence

$$m = n = r = 0.$$

Under this condition, observing  $b_2 \in p^{-1}\mathbb{Z}_p$ , we conclude that  $s + t \in \mathbb{Z}$ . Thus the contribution is calculated as

$$c \sum_{y=0}^{p-1} \beta\left(\begin{bmatrix} 0 & -1 \\ p^2 & 0 \end{bmatrix} \begin{bmatrix} 1 & \frac{y}{p} \\ 0 & 1 \end{bmatrix}\right) \beta\left(\begin{bmatrix} 0 & -1 \\ p^2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -\frac{y}{p} \\ 0 & 1 \end{bmatrix}\right) \quad (3.8)$$

with  $c = \text{vol}(\Gamma_0(p^2)_p \times \Gamma_0(p^2)_p)$ . Since  $\varepsilon(\frac{1}{2}, \pi_{1,p}) = 1$ , the eigenvalue of  $\beta$  for the Atkin-Lehner operator is 1 and

$$\beta\left(\begin{bmatrix} 0 & -1 \\ p^2 & 0 \end{bmatrix} \begin{bmatrix} 1 & \frac{y}{p} \\ 0 & 1 \end{bmatrix}\right) = \beta\left(\begin{bmatrix} 1 & 0 \\ py & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ p^2 & 0 \end{bmatrix}\right) = \beta\left(\begin{bmatrix} 1 & 0 \\ py & 1 \end{bmatrix}\right).$$

Let  $\mathcal{W}(\pi_{1,p}, \psi_p)$  be the Whittaker model of  $\pi_{1,p}$  with respect to  $\psi_p$ . We define a mapping

$$C : \mathcal{W}(\pi_{1,p}, \psi_p) \ni w(g) \longrightarrow w\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} g\right) = w\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} g \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right) \in \mathcal{W}(\pi_{1,p}, \bar{\psi}_p).$$

By the local newform theory for  $\text{GL}(2)$ , the dimension of the subspace of right  $\Gamma_0(p^2)_p$ -invariant vectors in  $\mathcal{W}(\pi_{1,p}, \bar{\psi}_p)$  is one. Hence,

$$C(\beta) = \bar{\beta}.$$

Now, (3.8) is calculated as

$$\begin{aligned} & c \sum_{y=0}^{p-1} \beta\left(\begin{bmatrix} 1 & 0 \\ py & 1 \end{bmatrix}\right) \beta\left(\begin{bmatrix} 1 & 0 \\ -py & 1 \end{bmatrix}\right) = c \sum_{y=0}^{p-1} \beta\left(\begin{bmatrix} 1 & 0 \\ py & 1 \end{bmatrix}\right) \beta\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ py & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right) \\ & = c \sum_{y=0}^{p-1} \beta\left(\begin{bmatrix} 1 & 0 \\ py & 1 \end{bmatrix}\right) C(\beta)\left(\begin{bmatrix} 1 & 0 \\ py & 1 \end{bmatrix}\right) = c \sum_{y=0}^{p-1} \beta\left(\begin{bmatrix} 1 & 0 \\ py & 1 \end{bmatrix}\right) \bar{\beta}\left(\begin{bmatrix} 1 & 0 \\ py & 1 \end{bmatrix}\right) > 0. \end{aligned}$$

Hence the total contribution of this type is some positive number. Combining the contribution of type I), we conclude

$$W_p(1) \neq 0.$$

For each  $i \in \{0, 1, 2, 3, 4\}$  let

$$\varphi_p^{\text{lev}, \chi^i}(x_1, x_2) = \chi^{-i}(\det(p^2 x_2)) \text{Ch}\left(\begin{bmatrix} \mathbb{Z}_p & p^{-1}\mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{bmatrix} \oplus \begin{bmatrix} p^{-1}\mathbb{Z}_p^\times & p^{-1}\mathbb{Z}_p \\ \mathbb{Z}_p & p^{-1}\mathbb{Z}_p^\times \end{bmatrix}\right).$$

Similar to (3.5),  $\varphi_p^{\text{lev}, \chi^i}$  is right  $K(p)_p^{\text{lev}} \times ((\Gamma_0(p^2)_p \times \Gamma_0(p^2)_p) \cap H^1(\mathbb{Q}_p))$ -invariant. For the pair of  $(\chi^i \pi_1, \chi^i \pi_2)$  with  $1 \leq i \leq 4$ , the proof of the nonvanishing of the local Whittaker function at  $p$  is similar to above. Thus,

**THEOREM 3.2.** *Let  $\pi_1 = \pi(\nu), \pi_2 = \pi(\nu^3)$  be the unitary irreducible cuspidal automorphic representations of  $\text{GL}_2(\mathbb{A})$  associated to the größencharacter  $\nu = \frac{\mu}{|\mu|}$ . Then, each of five globally generic*

endoscopic lift  $\chi^i \Theta(\pi_1 \boxtimes \pi_2) = \Theta(\chi^i \pi_1 \boxtimes \chi^i \pi_2)$  for  $0 \leq i \leq 4$  has a right  $K(p)_{\mathbb{A}}^{\text{lev}}$ -invariant automorphic form.

REMARK 3.3. For a locally generic admissible irreducible representation  $\tau$  of  $\text{GSp}_2(\mathbb{Q}_v)$ , Novodvorsky [24] defined a  $L$ -function of  $\tau$ . It holds

$$L(s, \chi_v^i \Theta(\pi_1 \boxtimes \pi_2)_v) = L(s, \mu \cdot \chi^i \circ N_{K/\mathbb{Q}})_v L(s, \mu^3 \cdot \chi^i \circ N_{K/\mathbb{Q}})_v.$$

REMARK 3.4. We give a Schwartz-Bruhat function  $\varphi_\infty \in \mathcal{S}(\text{M}_2(\mathbb{R})^2)$  for the  $\theta$ -lift of  $(\pi_1, \pi_2)$  as follows. Set

$$P_+(x) = \text{Tr}(x \begin{bmatrix} -\sqrt{-1} & -1 \\ -1 & \sqrt{-1} \end{bmatrix}), \quad P_-(x) = \text{Tr}(x \begin{bmatrix} \sqrt{-1} & 1 \\ -1 & \sqrt{-1} \end{bmatrix})$$

so that  $P_\pm(\rho(u_{t_1}, u_{t_2})x) = e^{-\sqrt{-1}(t_2 \pm t_1)} P_\pm(x)$  where  $u_{t_i} = \begin{bmatrix} \cos t_i & \sin t_i \\ -\sin t_i & \cos t_i \end{bmatrix} \in \text{SO}_2(\mathbb{R})$  for  $i = 1, 2$ .

Let  $s_1, s_2$  be indeterminants. Define  $\varphi_{\infty_j} \in \mathcal{S}(\text{M}_2(\mathbb{R})^2) \otimes \mathbb{C}[s_1, s_2]$  by

$$\varphi_\infty(x_1, x_2) = \exp(-\pi \left( \sum_{i=1}^2 a_i^2 + b_i^2 + c_i^2 + d_i^2 \right)) P_+(s_1 x_1 + s_2 x_2)^3 P_-(s_2 x_1 - s_1 x_2)$$

where we write  $x_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix}$ .

REMARK 3.5. The local  $\theta$ -lift  $\chi^i \theta(\pi_{1,p} \boxtimes \pi_{2,p})$  is the irreducible constituent denoted by  $\tau(S, \chi^i \pi(\mu)_p)$  of  $1 \rtimes \chi^i \pi_{1,p}$  in [33]. In particular, the central character  $\theta(\pi_{1,p} \boxtimes \pi_{2,p})$  is trivial. According to Roberts, Schmidt [33],  $\tau(S, \pi(\mu)_p)$  has a right  $K(p^4)_p$ -invariant Whittaker function, which is the newform. It is really realized by the  $\theta$ -lift as before with using  $\beta \in \mathcal{W}(\pi_{1,p}, \psi)$  and the following Schwartz-Bruhat function at  $p$ :

$$\varphi_p^{\text{para}}(x_1, x_2) = \text{Ch} \left( \begin{bmatrix} p\mathbb{Z}_p & p^{-1}\mathbb{Z}_p \\ p^3\mathbb{Z}_p & p\mathbb{Z}_p \end{bmatrix} \oplus p^{-1}\text{M}_2(\mathbb{Z}_p) \right).$$

#### 4. A comparison of $X$ and $\mathcal{A}_{1,11}^{\text{lev}}$

In this section, we shall discuss the relation of differential forms on  $X$  and  $\mathcal{A}_{1,11}^{\text{lev}}$  and of  $L$ -function of these varieties.

Recall the notations of Section 1. Let  $\Gamma' := \Gamma(11)$  be the congruence subgroup of level 11 in  $\text{Sp}_2(\mathbb{Z})$  which is normal in  $gK(11)^{\text{lev}}g^{-1}$ . We denote by  $G$  the quotient group  $gK(11)^{\text{lev}}g^{-1}/\Gamma'$ . Since  $G$  is finite, the restriction map induces an isomorphism of group cohomologies:  $H^3(K(11)^{\text{lev}}, \mathbb{C}) \simeq H^3(\Gamma', \mathbb{C})^G$ . Since  $K(11)^{\text{lev}} \backslash \mathbb{H}_2$  (resp.  $\Gamma' \backslash \mathbb{H}_2$ ) is an Eilenberg-MacLane space of  $K(11)^{\text{lev}}$  (resp.  $\Gamma'$ ), we have  $H^3(K(11)^{\text{lev}}, \mathbb{C}) = H^3(\mathcal{A}_{1,11}^{\text{lev}}, \mathbb{C})$  (resp.  $H^3(\Gamma', \mathbb{C}) = H^3(S_{\Gamma'}, \mathbb{C})$ ,  $S_{\Gamma'} := \Gamma' \backslash \mathbb{H}_2$ ) even if  $K(11)^{\text{lev}}$  has torsion elements because we are considering the complex coefficient. We note that  $S_{\Gamma'}$  is a quasi-projective smooth variety since  $\Gamma'$  is torsion free. Let  $\widetilde{S}_{\Gamma'}$  be a toroidal compactification of  $S_{\Gamma'}$  and let  $j : S_{\Gamma'} \hookrightarrow \widetilde{S}_{\Gamma'}$  be the natural inclusion. We consider the parabolic cohomology  $H_!^3(S_{\Gamma'}, \mathbb{C}) := \text{Im}(H^3(\widetilde{S}_{\Gamma'}, \mathbb{C}) \xrightarrow{j^*} H^3(S_{\Gamma'}, \mathbb{C}))$ . Then by observation in Section 7 in [26], we have  $H_{\text{cusp}}^3(S_{\Gamma'}, \mathbb{C}) = H_!^3(S_{\Gamma'}, \mathbb{C})$ . Here the cuspidal part  $H_{\text{cusp}}^3(S_{\Gamma'}, \mathbb{C})$  (resp.  $H_{\text{cusp}}^3(\mathcal{A}_{1,11}^{\text{lev}}, \mathbb{C})$ ) of  $H^3(S_{\Gamma'}, \mathbb{C})$  (resp.  $H^3(\mathcal{A}_{1,11}^{\text{lev}}, \mathbb{C})$ ) is given in terms of the  $(\mathfrak{g}, K)$ -cohomology (cf. Section 2 in [26]). That is a complement of the Eisenstein part in  $H^3(S_{\Gamma'}, \mathbb{C})$  (resp.  $H^3(\mathcal{A}_{1,11}^{\text{lev}}, \mathbb{C})$ ). Combining these, we have

$$H_{\text{cusp}}^3(\mathcal{A}_{1,11}^{\text{lev}}, \mathbb{C}) = H_!^3(S_{\Gamma'}, \mathbb{C})^G$$



and

$$H_{\text{cusp}}^3(\mathcal{A}_{1,11}^{\text{lev}}, \mathbb{C}) = m(\omega_2, K(11)^{\text{lev}})H^{2,1}(\mathfrak{g}, K; H_2) \oplus m(\omega_3, K(11)^{\text{lev}})H^{1,2}(\mathfrak{g}, K; H_3)$$

by the decomposition (2) at p.505 of [26] (see loc.cit. for the notation appears here). It is easy to see that  $\text{Gr}_3^W H^3(\mathcal{A}_{1,11}^{\text{lev}}, \mathbb{C})$  contains  $H_!^3(S_\Gamma, \mathbb{C})^G = H_{\text{cusp}}^3(\mathcal{A}_{1,11}^{\text{lev}}, \mathbb{C})$ . We hope that the equality  $\text{Gr}_3^W H^3(\mathcal{A}_{1,11}^{\text{lev}}, \mathbb{C}) = H_{\text{cusp}}^3(\mathcal{A}_{1,11}^{\text{lev}}, \mathbb{C})$ , but we do not know if it holds.

We now give a proof of Theorem 1.3.

*Proof.* The assertion (i) directly follows from the results in Section 2 and Section 3. We give a proof of (ii). Hereafter  $H^*$  means étale cohomology and we use freely the facts of étale cohomology (we refer [20] for this). Since  $X$  and  $\mathcal{A}_{1,11}^{\text{lev}}$  is birational to each other, we have a common non-empty open subvariety  $U$  defined over  $\mathbb{Q}$ . Now we have exact sequences of compact support étale cohomology:

$$\begin{aligned} \cdots \longrightarrow H_c^3(U_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \longrightarrow H_c^3(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) &= H^3(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \longrightarrow H_c^3((X \setminus U)_{\overline{\mathbb{Q}}}^{\text{red}}, \mathbb{Q}_\ell) \longrightarrow \cdots \\ \cdots \longrightarrow H_c^3(U_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \longrightarrow H_c^3(\mathcal{A}_{1,11}^{\text{lev}}_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) &\longrightarrow H_c^3((\mathcal{A}_{1,11}^{\text{lev}} \setminus U)_{\overline{\mathbb{Q}}}^{\text{red}}, \mathbb{Q}_\ell) \longrightarrow \cdots \end{aligned}$$

Here  $(X \setminus U)^{\text{red}}$  is the closed subscheme  $X \setminus U$  with the reduced scheme structure (it is same for  $(\mathcal{A}_{1,11}^{\text{lev}} \setminus U)^{\text{red}}$ ). The difference between  $H^3(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$  and  $H_c^3(\mathcal{A}_{1,11}^{\text{lev}}_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$  are described in terms of  $H_c^3((X \setminus U)_{\overline{\mathbb{Q}}}^{\text{red}}, \mathbb{Q}_\ell)$  and  $H_c^3((\mathcal{A}_{1,11}^{\text{lev}} \setminus U)_{\overline{\mathbb{Q}}}^{\text{red}}, \mathbb{Q}_\ell)$ . If the closed subschemes  $X \setminus U$  and  $\mathcal{A}_{1,11}^{\text{lev}} \setminus U$  contains a scheme  $Z$  of dimension less than or equal to one, then the cohomology  $H_c^3(Z_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$  vanishes. So we may assume that these subschemes are unions of surfaces (hence are of dimension 2). By Poincare duality, we have  $H_c^3((\mathcal{A}_{1,11}^{\text{lev}} \setminus U)_{\overline{\mathbb{Q}}}^{\text{red}}, \mathbb{Q}_\ell) \simeq H^1((\mathcal{A}_{1,11}^{\text{lev}} \setminus U)_{\overline{\mathbb{Q}}}^{\text{red}}, \mathbb{Q}_\ell)(-1)$ . Therefore, for a sufficiently large  $p \neq \ell$ , any eigenvalue of the Frobenius element  $\text{Frob}_p$  acting on  $H_c^3((\mathcal{A}_{1,11}^{\text{lev}} \setminus U)_{\overline{\mathbb{Q}}}^{\text{red}}, \mathbb{Q}_\ell)$  is of form  $p\alpha$  where  $\alpha \in \overline{\mathbb{Z}}$  is some Weil number. Since  $X$  is smooth cubic threefold, the same thing occurs for  $H^3(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$  as in Proposition 2.5. From this, any eigenvalue of the action of  $\text{Frob}_p$  on  $H_!^3(\mathcal{A}_{1,11}^{\text{lev}}_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$  is a multiple of  $p$ . This claims us that  $L(s, g)$  does not occur in  $L(s, H_!^3(\mathcal{A}_{1,11}^{\text{lev}}_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell))$ .

## 5. Some Remarks.

Keep the notations in Section 1 which is used to state Conjecture 1.5. Let  $V_\Gamma := \text{Gr}_3^W H_{\text{bett}}^3(S_{\Gamma\mathbb{C}}, \mathcal{L}_{a,b})$ . Assume

$$h^{a+b+3,0}(V_\Gamma) = h^{0,a+b+3}(V_\Gamma) = 0, \quad h^{a+2,b+1}(V_\Gamma) = h^{b+1,a+2}(V_\Gamma) \neq 0. \quad (5.1)$$

Our moduli space  $\mathcal{A}_{1,11}^{\text{lev}}$  is an example of such a variety for  $a = b = 0$ .

Let  $\Pi$  be an irreducible cuspidal automorphic representation of  $\text{GSp}_2(\mathbb{A})$  which arises from a non-holomorphic differential form in  $H^{a+2,b+1}(V_\Gamma)$ . By the argument before Conjecture 1.5, we guess that  $\Pi$  should be a weak endoscopic lift associated to a pair  $(\pi_1, \pi_2)$  so that  $\pi_{1,\infty}|_{\text{SL}_2}$  (resp.  $\pi_{2,\infty}|_{\text{SL}_2}$ ) is a discrete series representation of lowest weight  $a - b + 2$  (resp.  $a + b + 4$ ). We will consider when  $V_\Gamma$  tends to have the Hodge type (5.1). If  $\Theta(\pi_1^{\text{B}} \boxtimes \pi_2^{\text{B}})$  for a quaternion algebra  $B$  contributes to  $H^{a+2,b+1}(V_\Gamma)$ , then  $\Theta(\pi_1^{\text{B}} \boxtimes \pi_2^{\text{B}})$  has a right  $\Gamma(\mathbb{A})$ -invariant vector and  $B_\infty$  is split. On the other hand, if  $B_\infty$  is not split (i.e.,  $B$  is a definite quaternion algebra), then  $\Theta(\pi_1^{\text{B}} \boxtimes \pi_2^{\text{B}})$  is the so-called Yoshida lift and holomorphic. In that case, by the Hasse principle, the definite quaternion algebra  $B$  is ramified at some nonarchimedean place  $v$ . Here, we should remark that there is no Yoshida lift associated to  $(\pi_1, \pi_2)$ , if  $\pi_1^{\text{B}}$  and  $\pi_2^{\text{B}}$  does not exist simultaneously for a common  $B$  (i.e., one of  $\pi_{1,v}, \pi_{2,v}$  is a principal series representation for every nonarchimedean place  $v$ ).

We say  $\Gamma$  is *inadmissible*, if the Weil representation  $r_v'^2$  of  $\text{Sp}_2(\mathbb{Q}_v) \times \text{O}(D_v)$  for some nonarchimedean place  $v$  does not have a right  $\Gamma_v$ -invariant vector, where  $D_v$  is the unique division quater-

nion algebra  $D_v$  over  $\mathbb{Q}_v$ . We see below that  $K(N)$  and  $K(N)^{\text{lev}}$  are inadmissible for any positive integer  $N$ . For a place  $v$ , let  $r'_v{}^1$  (resp.  $r'_v{}^2$ ) be the Weil representation with respect to some nontrivial additive character  $\psi_v$  of  $\text{SL}_2(\mathbb{Q}_v) \times \text{O}(D_v)$  (resp.  $\text{Sp}_2(\mathbb{Q}_v) \times \text{O}(D_v)$ ) on  $\mathcal{S}(D_v)$  (resp.  $\mathcal{S}(D_v^2)$ ). It is easy to see that  $r'_v{}^1|_{\text{SL}(2)}$  does not have a nontrivial right  $\text{SL}_2(\mathbb{Z}_v)$ -invariant vector for a nonarchimedean place  $v$ . Consider the embedding from  $\text{SL}_2(\mathbb{Z}_v)$  into  $K(N)_v^{\text{lev}}$  via

$$\text{SL}_2 \ni \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \text{Sp}_2. \quad (5.2)$$

If  $r'_v{}^2|_{\text{Sp}_2}$  has a right  $K(N)_v^{\text{lev}}$ -invariant (or  $K(N)^{\text{lev}}$ -invariant) vector, then  $r'_v{}^1|_{\text{SL}_2}$  has a nontrivial right  $\text{SL}_2(\mathbb{Z}_v)$ -invariant vector which gives a contradiction. Therefore  $r'_v{}^2|_{\text{Sp}_2}$  does not have a right  $K(N)_v^{\text{lev}}$ -invariant (or  $K(N)^{\text{lev}}$ -invariant) vector. Therefore we can conclude that there are no contributions of weak endoscopic lifts to  $H^{a+b+3,0}(V_\Gamma)$  for any inadmissible  $\Gamma$ , since there is no holomorphic weak endoscopic lift associated to  $(\pi_1, \pi_2)$  which has a right  $\Gamma(\mathbb{A})$ -invariant vector. Furthermore in some case (cf.  $\Gamma = K(11)^{\text{lev}}$  and  $a = b = 0$ ),  $V_\Gamma$  tends to have the Hodge type (5.1).

In [28], we gave a conjecture for holomorphic parts of Siegel threefolds. It can be generalized as follows. This is also along the vein of Arthur's conjecture ([2], [41]).

**CONJECTURE 5.1.** *Let  $\Gamma \subset \text{GSp}_2(\mathbb{Q})$  be an arithmetic subgroup. Suppose  $V_\Gamma = \text{Gr}_3^W H_{\text{bett}i}^3(S_{\Gamma\mathbb{C}}, \mathcal{L}_{a,b})$  has Hodge numbers:*

$$h^{a+b+3,0}(V_\Gamma) = h^{0,a+b+3}(V_\Gamma) \neq 0, \quad h^{a+2,b+1}(V_\Gamma) = h^{b+1,a+2}(V_\Gamma) = 0 \quad (5.3)$$

*with  $a \geq b \geq 0$ . Suppose  $\Pi$  associated to a component of  $H^{a+b+3,0}(V_\Gamma)$  and  $\Pi$  has multiplicity one (it is so when  $h^{a+b+3,0} = 1$ ). Then,  $\Pi$  is a holomorphic Saito-Kurokawa representation.*

This conjecture is true if  $\Gamma$  is inadmissible. Indeed, according to Proposition 1.5 of Weissauer [44],  $\Pi$  is concluded to be a CAP representation or a weak endoscopic lift. Since  $\Gamma$  is *inadmissible*,  $\Pi$  can not be a weak endoscopic lift by the above argument. According to Theorem 4.1 of Soudry [38], every CAP representation associated to a Klingen or Borel parabolically induced representation is given by a  $\theta$ -lift of an irreducible automorphic representation  $\tau$  of  $\text{GO}(L_\mathbb{A})$  for a quadratic field  $L$ . In case that  $L$  is a real quadratic field, every automorphic form  $f$  of the  $\theta$ -lift has a nonzero Fourier coefficient associated to  $T = {}^tT$  with  $\det T \in -d_L(\mathbb{Q}^\times)^2$ , where  $d_L$  is the discriminant of  $L$  and positive. Hence  $f$  is neither holomorphic nor anti-holomorphic. Therefore this CAP representation cannot contribute to the  $h^{a+b+3,0}, h^{0,a+b+3}$ -parts. In case that  $L$  is an imaginary quadratic field, by Theorem 6.13, 7.2 of Kashiwara, Vergne [15], the Blattner parameter of the CAP representation associated to  $\tau$  is  $(c+1, 1)$  or  $(c+2, 2)$  if the weight of  $\tau|_{\text{GSO}(L)}$  (identified with a Größencharacter of  $L$ ) is  $c$ . Hence this CAP representation does not contribute to the  $h^{a+b+3,0}$ -part. Thus,  $\Pi$  is a holomorphic Saito-Kurokawa representation.

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